ELSEVIER Journal of Geometry and Physics 32 (1999) 1–13

JOURNAL OF GEOMETRY AND PHYSICS

Affine connections on homogeneous hypercomplex manifolds

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Received 17 December 1998

Abstract

It is the aim of this work to study affine connections whose holonomy group is contained in $Gl(n, \mathbb{H})$. These connections arise in the context of hypercomplex geometry. We study the case of homogeneous hypercomplex manifolds and introduce an affine connection which is closely related to the Obata connection [M. Obata, Japan J. Math. 26 (1956) 43–77]. We find a family of homogeneous hypercomplex manifolds whose corresponding connections are not flat with holonomy contained in $Sl(n, \mathbb{H})$. We consider first the 4-dimensional case and determine all the 4-dimensional real Lie groups which admit integrable invariant hypercomplex structures. We describe explicitly the Obata connection corresponding to these structures and by studying the vanishing of the curvature tensor, we determine which structures are integrable, obtaining as a byproduct a self-dual, non-flat, Ricci flat affine connection on $\mathbb{R}^4$ admitting a simply transitive solvable group of affine transformations. This result extends to a family of hypercomplex manifolds of dimension $4n$, $n > 1$, considered in [M.L. Barberis, I.D. Miatello, Quart. J. Math. Oxford 47 (2) (1996) 389–404]. We also give a sufficient condition for the integrability of hypercomplex structures on certain solvable Lie algebras. © 1999 Elsevier Science B.A. All right reserved.

Subj. Class.: Differential geometry
1991 MSC: 53C15; 32C10; 53C56
Keywords: Affine connections; Homogeneous hypercomplex manifolds

1. Introduction

A *hypercomplex structure* on a $4n$-dimensional C^∞ manifold M is a pair $\{J_1, J_2\}$ of fibrewise endomorphisms of the tangent bundle TM of M satisfying:

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PII: S0393-0440(99)00019-4

$$J_\alpha^2 = -I, \quad \alpha = 1, 2, \qquad J_1 J_2 = -J_2 J_1, \tag{1.1}$$

$$N_\alpha \equiv 0, \quad \alpha = 1, 2, \tag{1.2}$$

where I is the identity on the tangent space T_pM of M at p for all p in M and N_α is the Nijenhuis tensor corresponding to J_α:

$$N_\alpha(X, Y) = [J_\alpha X, J_\alpha Y] - [X, Y] - J_\alpha([X, J_\alpha Y] + [J_\alpha X, Y])$$

for all X, Y vector fields on M. If we set $J_3 = J_1 J_2$ it is well known that (1.2) implies $N_3 \equiv 0$. When we require $\{J_1, J_2\}$ to satisfy only (1.1) it is called an almost hypercomplex structure.

The quaternionic space $\mathbb{H}^n$ provides the standard model of a hypercomplex structure. Consider the real coordinates $(x_1, y_1, v_1, w_1, \dots, x_n, y_n, v_n, w_n)$ corresponding to a point $(q_1, \dots, q_n)$ in $\mathbb{H}^n$, where $q_l = x_l + iy_l + jv_l + kw_l$, $1 \leqslant l \leqslant n$. Then,

$$J_1(\partial/\partial x_l) = \partial/\partial y_l, \quad J_1(\partial/\partial v_l) = \partial/\partial w_l, \qquad 1 \leqslant l \leqslant n, \quad J_1^2 = -I,$$
$$J_2(\partial/\partial x_l) = \partial/\partial v_l, \quad J_2(\partial/\partial y_l) = -\partial/\partial w_l, \qquad 1 \leqslant l \leqslant n, \quad J_2^2 = -I,$$

defines a hypercomplex structure on $\mathbb{H}^n$.

Let M and M' be two manifolds admitting hypercomplex structures $\{J_1, J_2\}$ and $\{J_1', J_2'\}$, respectively. A C^∞ map $f : M \to M'$ is called *hypercomplex* (with respect to $\{J_1, J_2\}$ and $\{J_1', J_2'\}$) if the differential df of f satisfies $\mathrm{d}f\, J_\alpha = J_\alpha'\, \mathrm{d}f, \alpha = 1, 2$, that is, $f : (M, J_\alpha) \to (M', J_\alpha')$ is holomorphic for $\alpha = 1, 2$.

Assume that a $4n$-dimensional manifold M admits an atlas of charts $\{(U_a, \varphi_a)\}$ such that the transition functions $\varphi_a \circ \varphi_b^{-1} : \varphi_b(U_a \cap U_b) \to \varphi_a(U_a \cap U_b)$ are hypercomplex with respect to the standard hypercomplex structure on $\mathbb{R}^{4n} = \mathbb{H}^n$ considered above. It turns out that, by transferring the standard hypercomplex structure from $\mathbb{R}^{4n}$ to M by means of these charts, we obtain a globally defined hypercomplex structure on M. A hypercomplex structure $\{J_1, J_2\}$ on M is called *integrable* when it is obtained in the way just described.

In contrast with Newlander–Nirenberg's result for the complex case (cf. [13], Theorem 1.1), not every hypercomplex structure is integrable: the hypercomplex structures on $SU(3)$ and $SO(6)/SU(2)$ constructed in [11] provide such an example (see Section 2). Moreover, it can be shown that a manifold admitting an integrable hypercomplex structure is necessarily affine; on the other hand the $K3$ surfaces are examples of non-affine manifolds which admit hypercomplex structures (cf. [6]).

Associated with every hypercomplex structure $\mathcal{H}$ on M there is a canonical torsion-free affine connection $\nabla^{\mathcal{H}}$. It is known that $\mathcal{H}$ is integrable if and only if $\nabla^{\mathcal{H}}$ is flat (cf. [1,15]). We will use this fact together with the results in [4] to obtain Theorem 3.6 (Section 3.1).

2. The Obata connection

We start recalling that, given an arbitrary affine connection ∇ on M, the torsion, curvature and Ricci tensor fields T, R and Ric are defined as follows:

$$T(U, V) = \nabla_U V - \nabla_V U - [U, V], \tag{2.1}$$

$$R(U, V) = [\nabla_U, \nabla_V] - \nabla_{[U,V]}, \tag{2.2}$$

$$Ric(U, V) = \mathrm{tr}(R(\cdot, U)V), \tag{2.3}$$

for U, V vector fields on M, where $\mathrm{tr}(R(\cdot, U)V)$ denotes the trace of the map $Z \to R(Z, U)V$. We call ∇ torsion-free when $T \equiv 0$. If ∇ is torsion-free and $R \equiv 0$ (resp. $Ric \equiv 0$) then ∇ is called flat (resp. Ricci flat).

Lemma 2.1. *Let ∇ be a torsion-free connection. Then Ric is symmetric if and only if $R(X, Y)$ is traceless for all vector fields X, Y on M.*

Proof. Since ∇ is torsion-free, the Bianchi identity together with the Jacobi identity gives

$$Ric(X, Y) = Ric(Y, X) - \mathrm{tr}(R(X, Y))$$

and the lemma follows. □

Every hypercomplex structure on M uniquely determines an affine torsion-free connection with respect to which the corresponding endomorphisms are parallel, as the following result states.

Theorem 2.2 (Ref. [15]). *Let $\mathcal{H} = \{J_1, J_2\}$ be an almost hypercomplex structure on M. Then M admits a unique affine connection $\nabla^{\mathcal{H}}$ such that $\nabla^{\mathcal{H}} J_\alpha = 0$, $\alpha = 1, 2$, with torsion tensor $T^{\mathcal{H}} = \frac{1}{6}\sum_{\alpha=1}^{3} N_\alpha$. Moreover, $\mathcal{H}$ is hypercomplex if and only if $\nabla^{\mathcal{H}}$ is torsion-free.*

The connection $\nabla^{\mathcal{H}}$ of the above theorem will be called the *Obata connection* throughout the paper. It is given by the following formula (cf. [2]):

$$\nabla_X^{\mathcal{H}} Y = \frac{1}{2}[X, Y] + \frac{1}{12}\sum_{(\alpha,\beta,\gamma)} J_\alpha([J_\beta X, J_\gamma Y] + [J_\beta Y, J_\gamma X]) \tag{2.4}$$

$$+ \frac{1}{6}\sum_{\alpha=1}^{3} J_\alpha([J_\alpha X, Y] + [J_\alpha Y, X]) + \frac{1}{12}\sum_{\alpha=1}^{3} N_\alpha(X, Y), \tag{2.5}$$

where $J_3 = J_1 J_2$, (α, β, γ) runs over the cyclic permutations of $(1, 2, 3)$ and X, Y are vector fields on M.

Let $\mathcal{H} = \{J_1, J_2\}$ be an almost hypercomplex structure on M. For each $p \in M$, let $\mathcal{Q}_p$ denote the Lie subalgebra of $End(T_pM)$ generated by the endomorphisms induced by $\{J_1, J_2\}$ on the tangent space T_pM. The centralizer of $\mathcal{Q}_p$ in $Gl(T_pM)$ is isomorphic to $Gl(n, \mathbb{H})$ (dim $M = 4n$). Using this identification it follows that the condition $\nabla^{\mathcal{H}} J_\alpha = 0$, $\alpha = 1, 2$, in Theorem 2.2 is equivalent to $Hol(\nabla^{\mathcal{H}}) \subset Gl(n, \mathbb{H})$, where $Hol(\nabla^{\mathcal{H}})$ is the holonomy group of $\nabla^{\mathcal{H}}$.

If $Ric^{\mathcal{H}}$ (resp. $R^{\mathcal{H}}$) denotes the Ricci tensor (resp. curvature tensor) of $\nabla^{\mathcal{H}}$, it is well known that when $\nabla^{\mathcal{H}}$ is torsion-free, $Ric^{\mathcal{H}}$ is skew-symmetric (cf. [2]). Therefore, as a corollary to Lemma 2.1 we obtain the following.

Corollary 2.3. *If $\mathcal{H}$ is hypercomplex, $\nabla^{\mathcal{H}}$ is Ricci flat if and only if $R^{\mathcal{H}}(X, Y)$ is traceless for all vector fields X, Y on M.*

The following result establishes a necessary and sufficient condition for a hypercomplex structure to be integrable, in terms of its Obata connection.

Theorem 2.4 (Ref. [15]). *A hypercomplex structure $\mathcal{H}$ is integrable if and only if $\nabla^{\mathcal{H}}$ is flat.*

3. The homogeneous case

Let $M = G/K$ be homogeneous, i.e. G and K are Lie groups with K closed in G. A hypercomplex structure $\{J_1, J_2\}$ on G/K is said to be (left) invariant (or homogeneous) if $\tau(g): G/K \to G/K, hK \to ghK$ is hypercomplex with respect to $\{J_1, J_2\}$ for every $g \in G$. In this case the Obata connection is G-invariant, i.e. $\tau(g)$ is an affine transformation for every $g \in G$. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K, respectively. Recall that G/K is called reductive when $\mathfrak{k}$ admits an $Ad(K)$-invariant complement in $\mathfrak{g}$, where Ad denotes the adjoint representation of G in $\mathfrak{g}$.

Existence of invariant hypercomplex structures on G/K with compact G was studied in [11]. It can be shown that in the reductive case, no invariant hypercomplex structure is integrable. In fact, Doi [7] proved that a reductive homogeneous space G/K with real semisimple G does not admit a torsion-free, flat, G-invariant affine connection. This obstruction, together with Theorem 2.4, implies the following.

Proposition 3.1. *Let $M = G/K$ be a reductive homogeneous space with real semisimple G admitting an invariant hypercomplex structure. Then the corresponding Obata connection is non-flat. In particular, no invariant hypercomplex structure on M is integrable.*

It follows from the above proposition that the invariant hypercomplex structures on $SU(3)$ and $SO(6)/SU(2)$ constructed in [11] are not integrable; that is, the corresponding Obata connections are not flat.

Recall that a reductive homogeneous space G/K admits two canonical invariant connections ∇ and ∇' such that ∇ is torsion-free and has the same geodesics as ∇' (in [14] these connections are referred to as the canonical connections of the first and second kind, respectively). We observe that the connection $\nabla^{\mathcal{H}}$ associated to an invariant hypercomplex structure $\mathcal{H}$ on G/K is different from ∇', but $\nabla^{\mathcal{H}}$ is rigid with respect to ∇', i.e. the tensor field $S(X, Y) = \nabla'_X Y - \nabla^{\mathcal{H}}_X Y$ is parallel with respect to ∇' (cf. [12]). In general, $\nabla^{\mathcal{H}}$ is also different from ∇ (see Remark 3.9).

In [4] we classify the invariant hypercomplex structures on 4-dimensional real Lie groups; that is, we consider the case $K = \{e\}$ and $\dim G = 4$. In Section 3.1 we shall compute the Obata connections associated to such structures in order to determine, by means of Theorem 2.4, the ones which are integrable.

A hypercomplex structure on a $4n$-dimensional real Lie algebra $\mathfrak{g}$ is a pair $\{J_1, J_2\}$ of endomorphisms of $\mathfrak{g}$ satisfying Eqs. (1.1) and (1.2) on $\mathfrak{g}$. If G is a Lie group with Lie algebra $\mathfrak{g}$ it follows, by left translating $\{J_1, J_2\}$, that there is a one-to-one correspondence between hypercomplex structures on $\mathfrak{g}$ and invariant hypercomplex structures on G. Using this correspondence, the Obata connection $\nabla^{\mathcal{H}}$ associated with an invariant hypercomplex structure $\mathcal{H} = \{J_1, J_2\}$ on G can be regarded as a $\mathfrak{g}$-valued bilinear form $\nabla^{\mathcal{H}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the following properties:

$$\nabla^{\mathcal{H}}_X Y - \nabla^{\mathcal{H}}_Y X = [X, Y], \tag{3.1}$$

$$[\nabla^{\mathcal{H}}_X, J_\alpha] = 0 \tag{3.2}$$

for X, Y in $\mathfrak{g}$, $\alpha = 1, 2$ (see also Eq. (2.4)). Eq. (3.2) says that the endomorphisms $\nabla^{\mathcal{H}}_X$, $X \in \mathfrak{g}$, belong to the centralizer of $\{J_1, J_2\}$ in $End(\mathfrak{g})$, which is isomorphic to $\mathfrak{gl}(n, \mathbb{H})$, the Lie algebra of $Gl(n, \mathbb{H})$. The following lemma will be useful when trying to determine whether $\nabla^{\mathcal{H}}$ is Ricci flat.

Lemma 3.2. *$\nabla^{\mathcal{H}}$ is Ricci flat if and only if* $\mathrm{tr}(\nabla^{\mathcal{H}}_X) = 0$ *for all X in $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$.*

Proof. In view of Corollary 2.3 we have to show that $R^{\mathcal{H}}$ is traceless if and only if $\mathrm{tr}(\nabla^{\mathcal{H}}_X) = 0$ for all X in $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. For U, V in $\mathfrak{g}$ we have

$$\mathrm{tr}(R^{\mathcal{H}}(U, V)) = \mathrm{tr}([\nabla^{\mathcal{H}}_U, \nabla^{\mathcal{H}}_V] - \nabla^{\mathcal{H}}_{[U,V]}) = -\mathrm{tr}(\nabla^{\mathcal{H}}_{[U,V]})$$

and now the lemma follows. □

Two hypercomplex structures $\{J_1, J_2\}$ and $\{J_1', J_2'\}$ on $\mathfrak{g}$ are said to be equivalent if there exists an automorphism ϕ of $\mathfrak{g}$ such that $\phi\, J_\alpha = J_\alpha'\, \phi$ for $\alpha = 1, 2$.

There is an action of $SO(3)$ on the set of hypercomplex structures on $\mathfrak{g}$ as follows: given $A = (a_{ij})$ in $SO(3)$ and $\mathcal{H} = \{J_1, J_2\}$ we define

$$A \cdot \mathcal{H} = \left\{ \sum_{\alpha=1}^{3} a_{1\alpha} J_\alpha, \sum_{\alpha=1}^{3} a_{2\alpha} J_\alpha \right\},$$

where $J_3 = J_1 J_2$. It is not hard to check that $A \cdot \mathcal{H}$ defines a hypercomplex structure on $\mathfrak{g}$, with $(\sum_{\alpha=1}^{3} a_{1\alpha} J_\alpha)(\sum_{\alpha=1}^{3} a_{2\alpha} J_\alpha) = \sum_{\alpha=1}^{3} a_{3\alpha} J_\alpha$.

With the above definitions we can now state the following lemma.

Lemma 3.3. *Let $\mathcal{H} = \{J_1, J_2\}$ and $\mathcal{H}' = \{J_1', J_2'\}$ be two hypercomplex structures on $\mathfrak{g}$.*
(i) *If $\mathcal{H}$ and $\mathcal{H}'$ lie in the same $SO(3)$-orbit then $\nabla^{\mathcal{H}} = \nabla^{\mathcal{H}'}$;*
(ii) *if $\mathcal{H}$ is equivalent to $\mathcal{H}'$ then $\nabla^{\mathcal{H}}$ is flat (resp. Ricci flat) if and only if $\nabla^{\mathcal{H}'}$ is.*

Proof. If $\mathcal{H}$ and $\mathcal{H}'$ lie in the same $SO(3)$-orbit, then, since J_α' is a linear combination of $\{J_1, J_2, J_1 J_2\}$, $[\nabla^{\mathcal{H}}_X, J_\alpha'] = 0$ for X in $\mathfrak{g}$, $\alpha = 1, 2$, so by uniqueness of the Obata connection we must have $\nabla^{\mathcal{H}'} = \nabla^{\mathcal{H}}$ and (i) follows.

Next assume there is an automorphism ϕ of $\mathfrak{g}$ such that $\phi J_\alpha = J'_\alpha \phi$ for $\alpha = 1, 2$. Let $R^{\mathcal{H}}$ (resp. $Ric^{\mathcal{H}}$) and $R^{\mathcal{H}'}$ (resp. $Ric^{\mathcal{H}'}$) denote the curvature (resp. Ricci) tensors corresponding to $\nabla^{\mathcal{H}}$ and $\nabla^{\mathcal{H}'}$, respectively. Now (ii) follows from the following identities:

$$\nabla^{\mathcal{H}'}_{\phi U} = \phi \nabla^{\mathcal{H}}_U \phi^{-1}, \tag{3.3}$$

$$R^{\mathcal{H}'}(\phi U, \phi V) = \phi R^{\mathcal{H}}(U, V) \phi^{-1}, \tag{3.4}$$

$$Ric^{\mathcal{H}'}(\phi U, \phi V) = Ric^{\mathcal{H}}(U, V), \tag{3.5}$$

for U, V in $\mathfrak{g}$. □

Let $\mathfrak{H}_{\mathfrak{g}}$ denote the set of equivalence classes of hypercomplex structures on $\mathfrak{g}$ under the equivalence relation defined above. An element of $\mathfrak{H}_{\mathfrak{g}}$ is denoted $[\mathcal{H}]$ where $\mathcal{H}$ is a hypercomplex structure on $\mathfrak{g}$. The action of $SO(3)$ pushes down to $\mathfrak{H}_{\mathfrak{g}}$: given A in $SO(3)$ and $[\mathcal{H}]$ in $\mathfrak{H}_{\mathfrak{g}}$ set $A \cdot [\mathcal{H}] = [A \cdot \mathcal{H}]$. This is easily seen to be a well defined action on $\mathfrak{H}_{\mathfrak{g}}$.

The proof of the following result is contained in the proofs of Theorems 3.1, 3.3 and 3.4 in [4].

Theorem 3.4. *Let $\mathfrak{g}$ be a 4-dimensional real Lie algebra admitting a hypercomplex structure. Then $SO(3)$ acts transitively on $\mathfrak{H}_{\mathfrak{g}}$.*

We proved in [4] that for the Lie algebras described in cases (III) and (V) below, the corresponding isotropy subgroup is $O(2)$ so that there is a bijection of $\mathfrak{H}_{\mathfrak{g}}$ onto the 2-dimensional real projective space $\mathbb{R}P^2$. For the remaining three isomorphism classes of Lie algebras, the isotropy subgroup is all of $SO(3)$ and therefore the space $\mathfrak{H}_{\mathfrak{g}}$ reduces to a single point.

Theorem 3.4 and Lemma 3.3 imply the following corollary.

Corollary 3.5. *Let G be a 4-dimensional real Lie group admitting an invariant hypercomplex structure. Then the following conditions on G are equivalent:*
(i) *G admits an invariant integrable hypercomplex structure;*
(ii) *every invariant hypercomplex structure on G is integrable.*

3.1. The 4-dimensional case

In what follows we will obtain the invariant integrable hypercomplex structures in dimension 4. In [4] we show that a 4-dimensional real Lie algebra admitting a hypercomplex structure is isomorphic to one of the following Lie algebras $\mathfrak{g}$:

(I) $\mathfrak{g}$ abelian;
(II) $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{so}(3)$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$ ($\dim \mathfrak{z} = 1$);
(III) $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{a}$, $\mathfrak{b}$ an abelian ideal, $\mathfrak{a}$ an abelian subalgebra with bases $\{X, Y\}$, $\{Z, W\}$, respectively, such that:

$$[X, Z] = X, \quad [Y, Z] = Y, \quad [X, W] = Y, \quad [Y, W] = -X.$$

(IV) $\mathfrak{g} = \mathbb{R}A \oplus \mathfrak{g}'$, where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is abelian and $ad_{A|\mathfrak{g}'} = I$;
(V) $\mathfrak{g} = \mathbb{R}A \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_2$, where $\mathfrak{g}' = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ is a Heisenberg algebra with centre $\mathfrak{a}_2$ and $ad_{A|\mathfrak{a}_i} = (i/2)I$, $i = 1, 2$, where I denotes the identity on the corresponding vector spaces.

Note that the additive group of the quaternions $\mathbb{H}$ has an abelian Lie algebra; the Lie algebra of the multiplicative group of non-zero quaternions $\mathbb{H}^*$ is of type (II) and the group of motions of $\mathbb{C}$, *Aff*($\mathbb{C}$), has a type (III) Lie algebra. Let us denote by $S_{\mathbb{R}}$ (resp. $S_{\mathbb{C}}$) the simply connected solvable Lie group with Lie algebra of type (IV) (resp. (V)). It is well known (cf. [10]) that $S_{\mathbb{R}}$ and $S_{\mathbb{C}}$ act simply transitively on the 4-dimensional real and complex hyperbolic spaces, respectively. Keeping this notation, we are to prove the following result.

Theorem 3.6. *Let G be a 4-dimensional real Lie group. Then:*
(i) *G admits an invariant integrable hypercomplex structure if and only if G is locally isomorphic to either* $\mathbb{H}$, $\mathbb{H}^*$, *Aff*($\mathbb{C}$) *or* $S_{\mathbb{R}}$.
(ii) *G admits a non-integrable invariant hypercomplex structure if and only if G is locally isomorphic to* $S_{\mathbb{C}}$.

Proof. The first step is to determine, for the Lie algebras listed above, which hypercomplex structures are integrable. From Corollary 3.5 we know that, given $\mathfrak{g}$, the integrability condition is independent of the structure we choose, so we shall fix an arbitrary structure $\mathcal{H}$ on $\mathfrak{g}$ and compute the curvature tensor $R^{\mathcal{H}}$ of $\nabla^{\mathcal{H}}$ for this single case. To carry out this computation we will use the uniqueness of $\nabla^{\mathcal{H}}$ subject to the conditions (3.1) and (3.2) instead of the cumbersome formula (2.4).

In case (I) a hypercomplex structure $\mathcal{H}$ on $\mathfrak{g}$ is obtained by fixing two endomorphisms $\{J_1, J_2\}$ satisfying (1.1) on $\mathfrak{g}$. Conditions (3.1) and (3.2) are trivially satisfied setting $\nabla \equiv 0$, then by uniqueness $\nabla^{\mathcal{H}} = \nabla \equiv 0$ and therefore this structure is integrable.

In case (II) let $\{Z, X, Y, W\}$ be a basis of $\mathfrak{z} \oplus \mathfrak{so}(3)$ such that $Z \in \mathfrak{z}$ and

$$[X, Y] = W, \qquad [Y, W] = X, \qquad [W, X] = Y.$$

We fix the following hypercomplex structure $\mathcal{H} = \{J_1, J_2\}$:

$$J_1 Z = X, \qquad J_1 Y = W, \qquad J_1^2 = -I,$$
$$J_2 Z = Y, \qquad J_2 W = X, \qquad J_2^2 = -I.$$

The centralizer of $\{J_1, J_2\}$ in $End(\mathfrak{g})$, which is isomorphic to $\mathfrak{gl}(1, \mathbb{H})$, has a basis $\{I, J_1', J_2', J_3'\}$ where I is the identity and

$$J_1' X = Z, \qquad J_1' Y = W, \qquad J_1'^2 = -I,$$
$$J_2' Y = Z, \qquad J_2' W = X, \qquad J_2'^2 = -I,$$

with $J_3' = J_1' J_2'$. Now we compute the Obata connection $\nabla^{\mathcal{H}}$. We know from (3.2) that the endomorphism $\nabla^{\mathcal{H}}_U$ of $\mathfrak{g}$ must be a linear combination of $\{I, J_1', J_2', J_3'\}$ for all U in $\mathfrak{g}$. The coefficients of $\nabla^{\mathcal{H}}_U$ are uniquely determined by Eq. 3.1. It is easy to verify that $\nabla^{\mathcal{H}}$ is given as follows:

$$\nabla^{\mathcal{H}}_Z = -\tfrac{1}{2}I, \qquad \nabla^{\mathcal{H}}_X = \tfrac{1}{2}J_1', \qquad \nabla^{\mathcal{H}}_Y = \tfrac{1}{2}J_2', \qquad \nabla^{\mathcal{H}}_W = \tfrac{1}{2}J_3'.$$

Now one can verify that the curvature tensor $R^{\mathcal{H}}$ vanishes identically, hence $\{J_1, J_2\}$ is integrable.

In case (III) we fix the following hypercomplex structure on $\mathfrak{g}$:

$$J_1X = -W, \qquad J_1Y = Z, \qquad J_1^2 = -I,$$
$$J_2X = Y, \qquad J_2Z = -W, \qquad J_2^2 = -I.$$

By similar arguments to those in case (II) one gets that (3.1) and (3.2) determine the Obata connection $\nabla^{\mathcal{H}}$ uniquely by

$$\nabla^{\mathcal{H}}_X = 0, \quad \nabla^{\mathcal{H}}_Y = 0, \quad \nabla^{\mathcal{H}}_Z = -I, \quad \nabla^{\mathcal{H}}_W = J,$$

where J is given by

$$JY = X, \qquad JW = Z, \qquad J^2 = -I.$$

It is easily checked that if $R^{\mathcal{H}}$ vanishes identically, then $\{J_1, J_2\}$ is integrable.

In case (IV) we consider a basis $\{X, Y, Z\}$ of $\mathfrak{g}'$ and we may assume $\{J_1, J_2\}$ is given by

$$J_1A = X, \qquad J_1Y = Z, \qquad J_1^2 = -I,$$
$$J_2A = Y, \qquad J_2Z = X, \qquad J_2^2 = -I.$$

We set

$$\nabla^{\mathcal{H}}_A = I, \qquad \nabla^{\mathcal{H}}_V = 0 \ \forall V \in \mathfrak{g}'$$

and one verifies at once that Eqs. (3.1) and (3.2) are satisfied. Thus $\nabla^{\mathcal{H}}$ is the Obata connection associated to $\{J_1, J_2\}$ and clearly $R^{\mathcal{H}} \equiv 0$, therefore $\{J_1, J_2\}$ is integrable.

The above calculations and Corollary 3.5 imply that every hypercomplex structure on $\mathfrak{g}$ is integrable in cases (I)–(IV). The proof of the theorem will therefore be complete if we show that the Obata connection arising from case (V) is non-flat. Let $\mathfrak{a}_2 = \mathbb{R}Z$ be the center of $\mathfrak{g}'$ and $\{X, Y\}$ a basis of $\mathfrak{a}_1$ such that $[X, Y] = -[Y, X] = Z$. We fix the following hypercomplex structure $\{J_1, J_2\}$ on $\mathfrak{g}$:

$$J_1A = Z, \qquad J_1Y = X, \qquad J_1^2 = -I,$$
$$J_2A = \tfrac{\sqrt{2}}{2}X, \qquad J_2Z = \tfrac{\sqrt{2}}{2}Y, \qquad J_2^2 = -I.$$

We compute the Obata connection using the same idea as in case (II). If I is the identity on $\mathfrak{g}$, let $\{I, J_1', J_2', J_3'\}$ be the following basis of the centralizer of $\{J_1, J_2\}$:

$$J_1'Z = A, \qquad J_1'Y = X, \qquad J_1'^2 = -I,$$
$$J_2'A = -\tfrac{\sqrt{2}}{2}X, \qquad J_2'Z = \tfrac{\sqrt{2}}{2}Y, \qquad J_2'^2 = -I$$

and $J_3' = J_1' J_2'$. Combining Eqs. (3.1) and (3.2) we obtain the Obata connection:

$$\nabla_A^{\mathcal{H}} = \tfrac{3}{4} I, \quad \nabla_Z^{\mathcal{H}} = \tfrac{1}{4} J_1', \quad \nabla_X^{\mathcal{H}} = -\tfrac{\sqrt{2}}{4} J_2', \quad \nabla_Y^{\mathcal{H}} = \tfrac{\sqrt{2}}{4} J_3'.$$

Now, observing that J_1' and J_3' anticommute and $J_3' J_1' = J_2'$, we compute $R^{\mathcal{H}}(Y, Z) = \frac{\sqrt{2}}{16} [J_3', J_1'] = \frac{\sqrt{2}}{8} J_2'$. Hence $\nabla^{\mathcal{H}}$ is not flat and therefore $\{J_1, J_2\}$ is not integrable. We conclude from Corollary 3.5 that no hypercomplex structure on $\mathfrak{g}$ is integrable. Observe that $\operatorname{tr}(\nabla_U^{\mathcal{H}}) = 0$ for all U in $[\mathfrak{g}, \mathfrak{g}]$, so Lemma 3.2 implies that $\nabla^{\mathcal{H}}$ is Ricci flat. □

The $\mathfrak{g}$-valued bilinear form $\nabla^{\mathcal{H}}$ of case (V) above extends to a left invariant torsion-free, non-flat, Ricci flat connection on $S_{\mathbb{C}}$. It follows from the results of [16] that this connection is self-dual. Observe that $S_{\mathbb{C}}$ acts simply transitively on $\mathbb{R}^4$ (cf. [10]), therefore $\nabla^{\mathcal{H}}$ can be transferred to $\mathbb{R}^4$ having $S_{\mathbb{C}}$ as a group of affine transformations. Since $\nabla^{\mathcal{H}}$ is non-flat, Ricci flat it follows that it is not projectively flat and it can be shown that $\nabla^{\mathcal{H}} R^{\mathcal{H}} \neq 0$, that is, $\nabla^{\mathcal{H}}$ is not locally symmetric. This paragraph can be summarized as follows.

Proposition 3.7. *$\mathbb{R}^4$ carries a torsion-free, self-dual, non-flat, Ricci flat affine connection having a simply transitive solvable group of affine transformations.*

Remark 3.8. *If G is locally isomorphic to either $\mathbb{H}^*$, $Aff(\mathbb{C})$, $S_{\mathbb{R}}$ or $S_{\mathbb{C}}$, and $\mathcal{H}$ is an invariant hypercomplex structure on G it follows that $\nabla^{\mathcal{H}}$ is not a metric connection, in other words, there is no left invariant metric on G having $\nabla^{\mathcal{H}}$ as its Levi-Civita connection (we do not require the metric to be positive definite). We prove this assertion for the case of $S_{\mathbb{C}}$ (the remaining cases have analogous proofs). If there existed a non-degenerate bilinear form $\langle , \rangle$ on $\mathfrak{g}$ having $\nabla^{\mathcal{H}}$ as its Levi-Civita connection, then we would have*

$$\langle \nabla_A^{\mathcal{H}} U, A \rangle = \langle [A, U], A \rangle \tag{3.6}$$

for all $U \in \mathfrak{g}$, where A is as in case (V) above. The left-hand side of (3.6) is equal to $\frac{3}{4}\langle U, A \rangle$ and the right-hand side is 0, $\frac{1}{2}\langle U, A \rangle$ or $\langle U, A \rangle$ depending on $U = A$, U in $\mathfrak{a}_1$ or U in $\mathfrak{a}_2$, respectively. This implies $\langle A, \mathfrak{g} \rangle = 0$, which is impossible since $\langle , \rangle$ was assumed to be non-degenerate.

The above remark shows that our examples are in sharp contrast with the case when the Obata connection has non-degenerate Ricci tensor and zero Weyl curvature. Indeed, according to results of Alekseevsky and Marchiafava [2] in the latter case the Obata connection is the Levi-Civita connection of a pseudo-quaternionic Kähler metric.

Remark 3.9. *Given an arbitrary real Lie algebra $\mathfrak{g}$ let ∇ denote the following bilinear form on $\mathfrak{g}$: $\nabla_X Y = \frac{1}{2}[X, Y]$ (this is the canonical torsion-free connection, or (0)-connection; cf. [8]). Then the curvature and Ricci tensors corresponding to ∇ are easily seen to be equal to*

$$R(X, Y) = -\tfrac{1}{4}\, ad_{[X,Y]}, \qquad Ric(X, Y) = -\tfrac{1}{4}\, B(X, Y),$$

where B denotes the Killing form on $\mathfrak{g}$. *Observe that* ∇ *is flat if and only if* $\mathfrak{g}' \subset \mathfrak{z}$ *(or, equivalently, if and only if* $\mathfrak{g}$ *is 2-step nilpotent) and it is Ricci flat if and only if* $B \equiv 0$. *Therefore* ∇ *is neither flat nor Ricci flat in cases* (II)–(V) *above. In particular, the connections associated to hypercomplex structures on those Lie algebras cannot be equivalent to* ∇.

4. A distinguished $Gl(n, \mathbb{H})$-connection

In this section we introduce an affine connection associated with a special class of hypercomplex structures and relate it with the Obata connection (Theorem 4.1).

We say that an almost hypercomplex structure $\{J_1, J_2\}$ on $\mathfrak{g}$ is *abelian* [9,17] when $[J_\alpha X, J_\alpha Y] = [X, Y]$ for all $X, Y \in \mathfrak{g}$, $\alpha = 1, 2$. Note that this condition is stronger than the vanishing of N_α, $\alpha = 1, 2$, so in particular $\{J_1, J_2\}$ is hypercomplex. Observe that if $\mathcal{H}$ is abelian then $[J_z X, J_z Y] = [X, Y]$ for all $z = (z_1, z_2, z_3)$ in the Euclidean 2-sphere S^2, where $J_z = \sum_{\alpha=1}^{3} z_\alpha J_\alpha$ and $J_3 = J_1 J_2$ as usual.

Theorem 4.1. *Let* $\mathcal{H} = \{J_1, J_2\}$ *be an almost hypercomplex structure on* $\mathfrak{g}$. *There exists a unique affine connection* $\nabla^{\mathcal{H},ab}$ *satisfying the following properties:*
(i) $\nabla^{\mathcal{H},ab} J_\alpha = 0$, $\alpha = 1, 2$,
(ii) $T^{\mathcal{H},ab}(X, Y) = 3T^{\mathcal{H}}(X, Y) + \frac{1}{2}\sum_{\alpha=1}^{3}([X, Y] - [J_\alpha X, J_\alpha Y])$ *for all* $X, Y \in \mathfrak{g}$.

Proof. Set

$$\nabla_X^{\mathcal{H},ab} = \frac{1}{2}\left(ad_X - \sum_{\alpha=1}^{3} J_\alpha\, ad_X\, J_\alpha\right), \tag{4.1}$$

where $ad_X(Y) = [X, Y]$ for all $X, Y \in \mathfrak{g}$ and $J_3 = J_1 J_2$. It follows that $\nabla^{\mathcal{H},ab}$ satisfies the required properties. The uniqueness follows from the fact that any connection satisfying (i) is entirely determined by its torsion. □

Set $A^{\mathcal{H}}(X, Y) = \sum_{\alpha=1}^{3}([X, Y] - [J_\alpha X, J_\alpha Y])$, so that $T^{\mathcal{H},ab} = 3T^{\mathcal{H}} + \frac{1}{2}A^{\mathcal{H}}$.

Corollary 4.2. *Let* $\mathcal{H} = \{J_1, J_2\}$ *be an almost hypercomplex structure on* $\mathfrak{g}$. *The following conditions are equivalent:*
(i) $\nabla^{\mathcal{H},ab}$ *is torsion-free;*
(ii) $A^{\mathcal{H}} \equiv 0$;
(iii) $\mathcal{H}$ *is abelian.*

Proof. Assume (i) holds. Then Theorem 2.2 implies that $\nabla^{\mathcal{H},ab} = \nabla^{\mathcal{H}}$ and (ii) follows from the above theorem.

If (ii) holds then

$$\begin{aligned} 3[X, Y] &= [J_1 X, J_1 Y] + [J_2 X, J_2 Y] + [J_3 X, J_3 Y], \\ 3[J_1 X, J_1 Y] &= [J_1 J_1 X, J_1 J_1 Y] + [J_2 J_1 X, J_2 J_1 Y] + [J_3 J_1 X, J_3 J_1 Y] \\ &= [X, Y] + [J_3 X, J_3 Y] + [J_2 X, J_2 Y] \end{aligned}$$

and therefore $3([X, Y] - [J_1X, J_1Y]) = [J_1X, J_1Y] - [X, Y]$; hence $[J_1X, J_1Y] = [X, Y]$ for all $X, Y \in \mathfrak{g}$. An analogous proof gives $[J_2X, J_2Y] = [X, Y]$, that is $\mathcal{H}$ is abelian.

We finally show that (iii) implies (i). If $\mathcal{H}$ is abelian then clearly $A^{\mathcal{H}} \equiv 0$ and, since $\mathcal{H}$ is hypercomplex, we also have $T^{\mathcal{H}} \equiv 0$; hence $T^{\mathcal{H},ab} \equiv 0$. □

The following corollary is a straightforward consequence of (4.1) and Lemma 3.2.

Corollary 4.3. *If $\mathcal{H}$ is abelian then $\nabla^{\mathcal{H}}$ is Ricci flat.*

Corollary 4.4. *Let $\mathfrak{g}$ be a Lie algebra with* $\dim [\mathfrak{g}, \mathfrak{g}] \leqslant 2$ *and $\mathcal{H}$ a hypercomplex structure on $\mathfrak{g}$. Then $\nabla^{\mathcal{H}}$ is Ricci flat.*

Proof. We proved in [5] that any hypercomplex structure on $\mathfrak{g}$ must be abelian and now the result follows from the previous corollary. □

5. Integrability of hypercomplex structures on solvable extensions of 2-step nilpotent Lie algebras

Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra, that is, $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{z}$ where $\mathfrak{z}$ is the center of $\mathfrak{n}$. Assume that $\mathfrak{n}$ admits a hypercomplex structure $\mathcal{H}$. The following result gives a sufficient condition for $\mathcal{H}$ to be integrable (see also [9]).

Proposition 5.1. *If $\mathfrak{z}$ is J_α-stable, $\alpha = 1, 2$, then $\mathcal{H}$ is integrable, that is, $\nabla^{\mathcal{H}}$ is flat.*

Proof. It follows from (2.4) that $\nabla^{\mathcal{H}}_Z = 0$ for all Z in $\mathfrak{z}$ and $\nabla^{\mathcal{H}}_X Y \in \mathfrak{z}$ for all X, Y in $\mathfrak{n}$. Since $\nabla^{\mathcal{H}}$ is torsion-free we also have $\nabla^{\mathcal{H}}_X|_{\mathfrak{z}} = 0$ for all X in $\mathfrak{n}$. It now follows that $R^{\mathcal{H}} \equiv 0$. □

Corollary 5.2. *Any abelian hypercomplex structure on $\mathfrak{n}$ is integrable.*

Proof. The corollary follows from the above proposition, since any abelian hypercomplex structure leaves $\mathfrak{z}$ invariant. □

The above corollary implies, in particular, that all of the structures constructed in [3] are integrable, hence the corresponding connections are flat.

We next consider a family of 3-step solvable extensions of 2-step nilpotent Lie algebras. Let $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ be a 2-step nilpotent Lie algebra and set $\mathfrak{s} = \mathbb{R}A \oplus \mathfrak{n}$. If we extend the Lie bracket to $\mathfrak{s}$ by $\mathrm{ad}_A|_{\mathfrak{z}} = I$ and $\mathrm{ad}_A|_{\mathfrak{v}} = \frac{1}{2}I$, then $\mathfrak{s}$ becomes a 3-step solvable Lie algebra.

Proposition 5.3. *Let $\mathcal{H} = \{J_1, J_2\}$ be a hypercomplex structure on $\mathfrak{s}$ leaving both $\mathbb{R}A \oplus \mathfrak{z}$ and $\mathfrak{v}$ stable and such that $J_\alpha A \in \mathfrak{z}$, $\alpha = 1, 2, 3$ ($J_3 = J_1J_2$). Then $\mathcal{H}$ is integrable.*

Proof. $\mathbb{R}A \oplus \mathfrak{z}$ is a totally geodesic hypercomplex subalgebra and it follows that $\nabla_A^{\mathcal{H}}|_{\mathbb{R}A\oplus\mathfrak{z}} = I$, $\nabla_A^{\mathcal{H}}|_{\mathfrak{v}} = \frac{1}{2}I$ and $\nabla_Z^{\mathcal{H}} = 0$ for all $Z \in \mathfrak{z}$. Since $\nabla^{\mathcal{H}}$ is torsion-free it follows that $\nabla_V^{\mathcal{H}}|_{\mathbb{R}A\oplus\mathfrak{z}} = 0$ for every $V \in \mathfrak{v}$. Eq. (2.4) implies that $\nabla_V^{\mathcal{H}} W \in \mathbb{R}A \oplus \mathfrak{z}$ for all $V, W \in \mathfrak{v}$. Using all these facts it is not hard to verify that $R(X, Y) = 0$ for all $X, Y \in \mathfrak{s}$. □

We exhibit in [5] hypercomplex structures satisfying the hypothesis of the above proposition.

On the other hand, the corresponding solvable extension of the Heisenberg algebra carries a non-integrable hypercomplex structure. This will follow from the 4-dimensional case (Theorem 3.6). Indeed, let $\mathfrak{n}$ be the $(2n+1)$-dimensional Heisenberg algebra (with odd n) and $\mathfrak{s}$ the solvable extension considered above. Fix a basis $\{Z, X_1, Y_1, \dots, X_n, Y_n\}$ of $\mathfrak{n}$ such that $[X_k, Y_k] = Z$. Set $\mathfrak{g} = \text{span}\{A, Z, X_1, Y_1\}$ and $\mathfrak{w} = \text{span}\{X_2, Y_2, \dots, X_n, Y_n\}$.

Proposition 5.4. *Let $\mathcal{H} = \{J_1, J_2\}$ be a hypercomplex structure on $\mathfrak{s}$ leaving both $\mathfrak{g}$ and $\mathfrak{w}$ stable. Then $\nabla^{\mathcal{H}}$ is non-flat, Ricci flat. In particular, $\mathcal{H}$ is not integrable.*

Proof. Since $\mathfrak{g}$ is the 4-dimensional Lie algebra of case (V) (Theorem 3.6) and it is a totally geodesic hypercomplex subalgebra of $\mathfrak{s}$, we have that the connection $\nabla^{\mathcal{H}}$ is given as follows on $\mathfrak{g}$, where J'_α, $\alpha = 1, 2, 3$, are as in Theorem 3.6:

$$\nabla_A^{\mathcal{H}}|_{\mathfrak{g}} = \tfrac{3}{4} I, \quad \nabla_Z^{\mathcal{H}}|_{\mathfrak{g}} = \tfrac{1}{4} J'_1, \quad \nabla_{X_1}^{\mathcal{H}}|_{\mathfrak{g}} = -\tfrac{\sqrt{2}}{4} J'_2, \quad \nabla_{Y_1}^{\mathcal{H}}|_{\mathfrak{g}} = \tfrac{\sqrt{2}}{4} J'_3,$$

and therefore $\nabla^{\mathcal{H}}$ is non-flat. Using Eq. (2.4) one can also check that

$$\nabla_A^{\mathcal{H}}|_{\mathfrak{v}} = \tfrac{1}{2} I, \qquad \nabla_{X_1}^{\mathcal{H}}|_{\mathfrak{w}} = \nabla_{Y_1}^{\mathcal{H}}|_{\mathfrak{w}} = \nabla_Z^{\mathcal{H}}|_{\mathfrak{w}} = 0,$$

$\nabla_V^{\mathcal{H}} W \in \mathfrak{g}$ and $\nabla_V^{\mathcal{H}}|_{\mathfrak{g}} = 0$ for all $V, W \in \mathfrak{w}$. It follows that $\text{tr}(\nabla_U^{\mathcal{H}}) = 0$ for all $U \in [\mathfrak{s}, \mathfrak{s}]$, then Lemma 3.2 implies that $\nabla^{\mathcal{H}}$ is Ricci flat. □

Remark 5.5. *A hypercomplex structure satisfying the hypothesis of the above proposition was constructed in* [5].

Acknowledgements

The author would like to thank Prof. I. Dotti Miatello for helpful suggestions and continuous encouragement and the University of California at San Diego for its hospitality during part of the preparation of the manuscript.

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